ON CHARACHTERIZING LINEAR LIE GROUPS BY VON NEUMANN'S THEOREM

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Let G be a closed topological subgroup of $GL_n(\mathbb{R})$, we shall, in this paper, characterize Lie groups, as closed linear topological groups, by Von Neumann's theorem, as smooth manifolds. This appeared first in David Hilbert's address entitled "Mathematical Problems" before the International Congress of Mathematicians in Paris in 1900, he proposed a list containing 23 problems varying over almost all branches of mathematics with the idea that their solutions would lead to progress in mathematics. Among these problems, the 5th, was about defining Lie groups as differentiable manifolds. The answer was given with the work of Andrew Gleason, Deane Montgomery and Leo Zippin in the 1950s.

We shall here give a simple proof of this result, based on Von Neumann's theorem, in addition to some results on differentiable and smooth manifolds.

1 The matrix exponential function

In the first section, we shall review some basic properties of the matrix exponential function. Recall that the exponential function is a matrix function defined on square matrices, analogy to the ordinary exponential function, given by the power series

$$\sum_{k\geq 0} \frac{1}{k!} A^k$$

for every matrix $A \in \mathbb{R}^{n\times n}$. Let $||.||$ be a chosen matrix norm on $\mathbb{R}^{n\times n}$.

Proposition 1.1. *1. The matrix exponential map is locally invertible : There exists an open neighberhood $V_0 \subset \mathbb{R}^{n\times n}$ and $V_{I_n} \subset GL_n(\mathbb{R})$ such that :*

$$\exp : V_0 \to V_{In} \textit{ is a diffeomorphism}$$

2. For all $A, B, C \in \mathbb{R}^{n\times n}$ such that $||A||, ||B||, ||C|| \leq \frac{1}{2}$ and $\exp(A)\exp(B) = \exp(C)$ we have that

$$||C - A - B|| \leq 17(||A|| + ||B||)^2$$

3. For all $A, B \in \mathbb{R}^{n\times n}$

$$\exp(A + B) = \lim_{k\to\infty} (\exp(\frac{A}{k}) \exp(\frac{B}{k}))^k$$

Proof. 1. as $\exp : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ is C^∞, even analytic, we see that $\forall H \in \mathbb{R}^{n\times n}$

$$\left|\left|\frac{exp(0+H) - \exp(0) - id_{\mathbb{R}^{n\times n}}(H)}{H}\right|\right| \leq ||H|| \sum_{k\geq 2} \frac{1}{k!} H^{k-2} \underset{||H||\to 0}{\longrightarrow} 0$$

Since $\sum \frac{1}{k!} H^{k-2} < \infty$. Hence $d_0(\exp) = id_{\mathbb{R}^{n\times n}}$ which is a bounded isomorphism. We apply Inverse function theorem, which gives the wanted result.
We can define $\log := \exp^{-1}$ as follow :

$$\forall X \in \mathbb{R}^{n\times n} : ||X|| < 1 \qquad \log(I_n - X) = -\sum_{k\geq 1} \frac{1}{k!} X^k$$

2. Pose $R := \exp(C) - C - I_n$ and $S := \exp(A)\exp(B) - I_n - A - B$. We get

$$||R|| = ||\sum_{k\geq 2} \frac{1}{k!} C^k|| \leq \sum_{k\geq 2} \frac{1}{k!} ||C||^k$$

as $||C|| \leq \frac{1}{2}$, for $k \geq 2$: $\quad ||C||^k \leq ||C||^2$. And hence

$$||R|| \leq ||C||^2 \sum_{k\geq 2} \frac{1}{k!} I_n = ||C||^2(\exp(I_n) - 2I_n) \leq ||C||^2$$

Same way

$$||S|| = ||\sum_{k\geq 2}\sum_{j=0}^{k} \frac{1}{j!}\frac{1}{(k-j)!} A^k B^{j-k}|| \leq \sum_{k\geq 2} \frac{1}{k!}(||A|| + ||B||)^k$$

$$\leq (||A|| + ||B||)^2 \sum_{k\geq 2} \frac{1}{k!} I_n = (||A|| + ||B||)^2(\exp(I_n) - 2I_n)$$

$$\leq (||A|| + ||B||)^2$$

Since $C = A + B + S - R$,

$$||C|| \leq ||A|| + ||B|| + (||A|| + ||B||)^2 + ||C||^2 \leq 2(||A|| + ||B||) + \frac{1}{2}||C||$$

(As $||A|| + ||B|| \leq 1 \Rightarrow (||A|| + ||B||)^2 \leq ||A|| + ||B||$) We get

$$\frac{1}{2}||C|| \leq 2(||A|| + ||B||) \Rightarrow ||C|| \leq 4(||A|| + ||B||)$$

Finally,

$$||C - A - B|| = ||S - R|| \leq (||A|| + ||B||)^2 + ||C||^2 \leq (||A|| + ||B||)^2 + 16(||A|| + ||B||)^2$$

$$\leq 17(||A|| + ||B||)^2$$

3. For a large $k \in \mathbb{N}$, the matrices

$$\frac{A}{k}, \frac{B}{k} \underset{k\to\infty}{\longrightarrow} 0_{\mathbb{R}^{n\times n}}$$

Hence, from 1. $\exists C_k \in \mathbb{R}^{n\times n}, C_k \underset{k\to\infty}{\longrightarrow} 0$ and

$$\exp(\frac{A}{k})\exp(\frac{B}{k}) = \exp(C_k) \underset{k\to\infty}{\longrightarrow} I_n$$

We may assume that $||\frac{A}{k}||, ||\frac{B}{k}||, ||C_k|| \leq \frac{1}{2}$. Then from 2.

$$||C_k - \frac{A}{k} - \frac{B}{k}|| \leq \frac{17}{k^2}(||A|| + ||B||)^2$$

$$\Rightarrow ||kC_k - A - B|| \leq \frac{17}{k}(||A|| + ||B||)^2 \underset{k\to\infty}{\longrightarrow} 0$$

Hence

$$kC_k \underset{k\to\infty}{\longrightarrow} A + B \;\; \Rightarrow \;\; \exp(kC_k) = \exp(C_k)^k \underset{k\to\infty}{\longrightarrow} \exp(A+B)$$

and thus,

$$(\exp(\frac{A}{k})\exp(\frac{B}{k}))^k \underset{k\to\infty}{\longrightarrow} \exp(A+B)$$

□

2 Lie algebra of a Lie group and main theorem

We define the Lie algebra of a linear Lie group (i.e. a closed topological subgroup G of $GL_n(\mathbb{R})$ as follow :

$$L(G) := \{A \in \mathbb{R}^{n\times n} : \exp(tA) \in G \forall t \in \mathbb{R}\}$$

Proposition 2.1. *$L(G)$ is an $\mathbb{R}$-subspace of $\mathbb{R}^{n\times n}$.*

Proof. Let $A, B \in L(G), \lambda \in \mathbb{R}$. We have from 3.

$$\exp(A + \lambda B) = \lim_{k\to\infty}(\exp(\frac{A}{k})\exp(\lambda\frac{B}{k}))^k$$

as $\exp(\frac{A}{k}), \exp(\lambda\frac{B}{k}) \in G$, this yields $(\exp(\frac{A}{k})\exp(\lambda\frac{B}{k}))^k \in G$ by the group structure of G. And since it's closed :

$$\exp(A + \lambda B) = \lim_{k\to\infty}(\exp(\frac{A}{k})\exp(\lambda\frac{B}{k}))^k \in G \;\; \Rightarrow A + \lambda B \in L(G)$$

□

Theorem 2.2 (Von Neumann)**.** *Let G be a closed subgroup of $GL_n(\mathbb{R})$, then there is an open neighbourhood U_0 in $L(G)$ and V_{I_n} in G such that*

$$\exp_{|U_0} : U_0 \longrightarrow V_{I_n} \textit{ is a diffeomorphism}$$

Proof. As seen below, since $L(G)$ is an $\mathbb{R}$-subspace of $\mathbb{R}^{n\times n}$, $L(G) := L$ admit a supplementary subspace S in $\mathbb{R}^{n\times n}$:

$$\mathbb{R}^{n\times n} = L \oplus S$$

Consider

$$\begin{array}{rcccc} f & : & \mathbb{R}^{n\times n} & \longrightarrow & GL_n(\mathbb{R}) \\ & & X = l + s & \longmapsto & \exp(l)\exp(s) \end{array}$$

It is easy to see that $id_{\mathbb{R}^{n\times n}}$, is a bounded isomorphism. We apply Inverse function theorem to get $U \in \mathcal{V}_0(\mathbb{R}^{n\times n})$, $V \in \mathcal{V}_{I_n}(GL_n(\mathbb{R}))$ such that

$$f : U \longrightarrow V \text{ is a diffeomorphism}$$

Pose $U = B_{\frac{1}{k}}(0) = \{X \in \mathbb{R}^{n\times n},\ ||X|| \leq \frac{1}{k}\}$. We have that

$$f(L \cap B_{\frac{1}{k}}) \subset G \cap f(B_{\frac{1}{k}})$$

We will show that those subsets are equal actually. Suppose they are not, then

$\exists x_k \underset{k\to\infty}{\longrightarrow} 0$ and $f(x_k) \in G \setminus f(L \cap B_{\frac{1}{k}})$.

As $x_k = l_k + s_k$, by continuity of the projection on S, $s_k \underset{k\to\infty}{\longrightarrow} 0$ and $s_k \neq 0$ because otherwise, $x_k \in L \cap B_{\frac{1}{k}}$.

Pose $\forall k \in \mathbb{N}$

$$r_k := \frac{s_k}{||s_k||}$$

Since $||r_k|| = 1$, $r_k \in B_{I_n}(0)$ the unit ball of $\mathbb{R}^{n\times n}$. As we are in finite dimensions, it is compact. We can hence extract a convergent subsequence from r_k.

Replacing it with this subsequence, we may that $r_k \underset{k\to\infty}{\longrightarrow} r$. We pose $\forall t \in \mathbb{R}$ $\frac{t}{||s_k||} = a + b_k$ with $a \in \mathbb{Z}$ and $|b_k| < \frac{1}{2}$, $b_k \in \mathbb{R}$.

We have then that

$$\exp(s_k)^a = \exp(as_k) = \exp(tr_k - b_k s_k) \underset{k\to\infty}{\longrightarrow} \exp(tr) \in G$$

(since $|b_k| < \infty$ and $s_k \underset{k\to\infty}{\longrightarrow} 0$)

Hence $r \in L \cap S = \{0\} \Rightarrow r = 0$ which is absurd since $||r|| = 1$. Thus

$$f(L \cap B_{\frac{1}{k}}) = G \cap f(B_{\frac{1}{k}}) \Rightarrow \exp_{|L} : B_{\frac{1}{k}} \in \mathcal{V}_0(L) \longrightarrow f(B_{\frac{1}{k}}) \in \mathcal{V}_{I_n}(G)$$

$\square$

3 Differentiable and smooth manifolds

Let M be a topological space, we say that M is a topological manifold of dimension n if M is Hausdorff, second countable and locally euclidean of dimension n : Each point of M has a neighbourhood that is homeomorphic to $\mathbb{R}^n$. More particularly, the third assertion means that for every $x \in M$ we can find :

- an open subset $U \subset M$ containing x.
- an open subset $U' \subset \mathbb{R}^n$.

- $\phi : U \longrightarrow U'$ a homemorphism.

Definition 3.1.
- *The pairs (U, ϕ) as described above are called* charts *of M.*
- *If $(U, \phi), (V, \psi)$ are two charts such that $U \cap V = \emptyset$, then the composition map $\psi \circ \phi^{-1} : \phi(U \cap V) \longrightarrow \psi(U \cap V)$ is called* ***the transition map*** *from ϕ to ψ.*
- *Two charts $(U, \phi), (V, \psi)$ are said to be* ***smoothly compatible*** *if $U \cap V = \emptyset$ or the transition map is a diffeomorphism.*
- *We define an atlas for M to be a collection of charts whose domains cover M, an Atlas is said to be smooth if each two charts in it are smoothly compatible.*

We define finally a smooth manifold as a topological manifold with a smooth atlas.
Recall Theorem 2.2, by posing $\forall g \in G$:

$$\begin{array}{cccc} \phi_g & : & \mathcal{V}_g = g\mathcal{V}_{I_n} & \longrightarrow & \mathcal{V}_0 \\ & & gx & \longmapsto & log(x) \end{array}$$

Then this is clearly a homeomorphism, from a g-neighbourhood of G onto a neighbourhood of $\mathbb{R}^n$ and thus $(\mathcal{V}_g, \phi_g)$ is a chart, hence, G is a topological manifold. By computing for $g, h \in G$

$$\phi_h \circ \phi_g^{-1} : \phi_g(\mathcal{V}_g \cap \mathcal{V}_h) \longrightarrow \phi_h(\mathcal{V}_g \cap \mathcal{V}_h)$$

Then clearly, the transition map between any two charts $(\mathcal{V}_g, \phi_g), (\mathcal{V}_h, \phi_h)$ is a diffeomorphism, and hence, G is a smooth manifold.